Bayesian Confirmation Measures and Their Properties – A New Perspective

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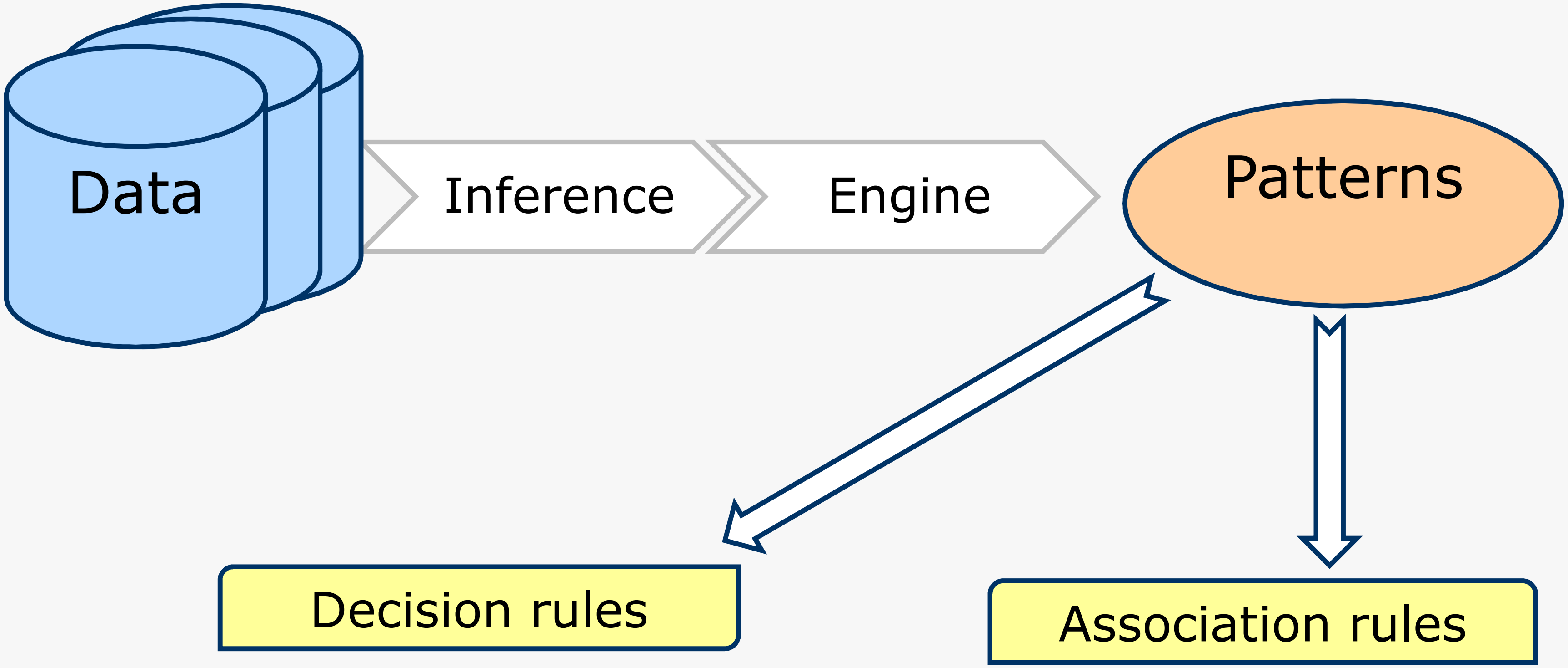
Nowe spojrzenie na Bayesowskie miary konfirmacji i ich właściwości

Rule induction



If symptom s_1 is present
and symptoms s_2
and s_3 are absent
then disease d_1

If symptom s_1 is present
then symptoms s_2
and s_3 are absent

Rule induction

- Patterns in form of rules are induced from a data table
- $S=\langle U, A\rangle$ – *data table*, where U and A are finite, non-empty sets
  U – *universe of objects*; A – set of *attributes*
- $S=\langle U, C, D\rangle$ – *decision table*, where C – set of *condition attributes*,
  D – set of *decision attributes*, $C\cap D=\varnothing$

- *Rule* induced from S is a *consequence relation*:
  E → H *read as* ***if E then H***
  where

  E is condition (evidence or premise) and
  H is conclusion (hypothesis or decision)
  formula built from attribute-value pairs (q,v)

Rule induction

Characterization of nationalities

U	*Height*	*Hair*	*Eyes*	*Nationality*	Support
1	*tall*	*blond*	*blue*	*Swede*	270
2	*medium*	*dark*	*hazel*	*German*	90
3	*medium*	*blond*	*blue*	*Swede*	90
4	*tall*	*blond*	*blue*	*German*	360
5	*short*	*red*	*blue*	*German*	45
6	*medium*	*dark*	*hazel*	*Swede*	45

C *D*

- E.g. decision rules induced from „characterization of nationalities”:
  1) If (*Height=tall*), then (*Nationality=Swede*)
  2) If (*Height=medium*) & (*Hair=dark*), then (*Nationality=German*)

Motivations

The number of rules
induced from datasets is usually quite large

- overwhelming for human comprehension,
- many rules are irrelevant or obvious
  (low practical value)

rule evaluation – interestingness (attractiveness) measures
(e.g. support, confidence, gain, rule interest, lift,
measures of Bayesian confirmation)

- each measure was proposed to capture different characteristics of rules
- the number of proposed measures is very large

Motivations

The choice of an interestingness measure for a certain application is a difficult problem

- there is no evidence which measure(s) is the best
- the users' expectations vary,
- the number of proposed measures is overwhelming

properties of interestingness measures, which reflect users' expectations towards the behavior of measures in particular situations

- property of monotonicity M (Greco, Pawlak & Słowiński 2004)
- Ex_1 property and its generalization to weak Ex_1
- property of logicality L and its generalization to weak L (Fitelson 2006; Crupi, Tentori & Gonzalez 2007 Greco, Słowiński & Szczęch 2012)
- …

need to analyze measures with respect to their properties

Presentation plan

- Property of confirmation and its different definitions
- Popular confirmation measures
- Properties of confirmation measures
  - Symmetry properties
  - Property of concordance
- Summary

Notation

- Used notation corresponding to a 2x2 contingency table of rule's premise and conclusion

  $a=sup(H,E)$ is the number of objects in U satisfying both the premise E and the conclusion H of a rule $\boldsymbol{E \rightarrow H}$,

  $b=sup(H, \neg E)$,

  $c=sup(\neg H, E)$,

  $d=sup(\neg H, \neg E)$,

  $a+c=sup(E)$,

  $a+b=sup(H)$,…

	H	$\neg H$	Σ
E	a	c	$a+c$
$\neg E$	b	d	$b+d$
Σ	$a+b$	$c+d$	$a+b+c+d=n$

- a, b, c and d can also be regarded as frequencies that can be used to estimate probabilities:
  e.g., $P(E)=(a+c)/n$, $P(H)=(a+b)/n$, $P(H|E) = a/(a+c)$.

Property of confirmation

Generally, measures possessing the property of confirmation (confirmation measures) are expected to obtain:

- values **>0** when the premise of a rule confirms the conclusion,
- values **= 0** when the rule's premise and conclusion are neutral to each other,
- values **< 0** when the premise disconfirms the conclusion.

- What does „premise confirms conclusion" mean?
- How to quantify such confirmation?

Property of confirmation

- Four definitions in the literature:
  - Bayesian confirmation
  - strong Bayesian confirmation: $P(H|E)>P(H|\neg E)$
  - likelihoodist confirmation: $P(E|H)>P(E)$
  - strong likelihoodist confirmation: $P(E|H)>P(E|\neg H)$
- An attractiveness measure $c(H,E)$, has the property of Bayesian confirmation if is satisfies the following condition:

$$c(H,E)\begin{cases} >0 \ \ if \ \ P(H|E)>P(H) \\ =0 \ \ if \ \ P(H|E)=P(H) \\ <0 \ \ if \ \ P(H|E)<P(H) \end{cases}$$

Property of confirmation

- Bayesian approach is related to the idea that the E confirms H, if H is more frequent with E rather than with $\neg E$ (perspective of rule's conclusion)

- Bayesian confirmation: $P(H|E)>P(H)$
  - H is satisfied more often when E is satisfied (then, this frequency is $P(H|E)$), rather than generically ($P(H)$)
    Assumtion: $P(E)\neq 0$

- strong Bayesian confirmation: $P(H|E)>P(H|\neg E)$
  - H is satisfied more often, when E is satisfied, rather than when *not* E is satisfied
    Assumtion: $P(E)\neq 0$, $P(\neg E)\neq 0$

Property of confirmation

- Likelihoodist approach is based on the idea that *E* confirms *H*, if *E* is more frequent with *H* rather than with $\neg H$ (perspective of rule's premise)
  - likelihoodist confirmation: $P(E|H)>P(E)$
  - strong likelihoodist confirmation: $P(E|H)>P(E|\neg H)$

Logical equivalance of four definitions of confirmation

- Bayesian confirmation: a/(a+c) > (a+b)/n
- strong Bayesian confirmation: a/(a+c) > b/(b+d)
- likelihoodist confirmation: a/(a+b)>(a+c)/n
- strong likelihoodist confirmation: a/(a+b) > c/(c+d)
- Obviously, the above definitions differ.
  - What is the relationship between them?
  - Do they „switch” (between +, zero and —) at the same times?
- All four definitions boil down to one general, always-defined formulation:

$$c(H,E)\begin{cases} >0 \ \ if \ \ ad - bc > 0 \\ =0 \ \ if \ \ ad - bc = 0 \\ <0 \ \ if \ \ ad - bc < 0 \end{cases}$$

Advantage: *ad-bc* is never undefined, no denominator

Popular confirmation measures

$$D(H,E) = \mathrm{P}(H \mid E) - \mathrm{P}(H) = \frac{a}{a+c} - \frac{a+b}{n} = \frac{ad-bc}{n(a+c)}$$

$$M(H,E) = \mathrm{P}(E \mid H) - \mathrm{P}(E) = \frac{a}{a+b} - \frac{a+c}{n} = \frac{ad-bc}{n(a+b)}$$

$$S(H,E) = \mathrm{P}(H \mid E) - \mathrm{P}(H \mid \neg E) = \frac{a}{a+c} - \frac{b}{b+d} = \frac{ad-bc}{(a+c)(b+d)}$$

$$N(H,E) = \mathrm{P}(E \mid H) - \mathrm{P}(E \mid \neg H) = \frac{a}{a+b} - \frac{c}{c+d} = \frac{ad-bc}{(a+b)(c+d)}$$

$$C(H,E) = \mathrm{P}(E \wedge H) - \mathrm{P}(E)\mathrm{P}(H) = \frac{a}{n} - \frac{(a+c)(a+b)}{n^2} = \frac{ad-bc}{n^2}$$

$$F(H,E) = \frac{\mathrm{P}(E \mid H) - \mathrm{P}(E \mid \neg H)}{\mathrm{P}(E \mid H) + \mathrm{P}(E \mid \neg H)} = \frac{\dfrac{a}{a+b} - \dfrac{c}{c+d}}{\dfrac{a}{a+b} + \dfrac{c}{c+d}} = \frac{ad-bc}{ad+bc+2ac}$$

Popular confirmation measures

$$D(H,E) = \mathrm{P}(H \mid E) - \mathrm{P}(H) = \frac{a}{a+c} - \frac{a+b}{n} = \frac{ad-bc}{n(a+c)}$$

- Notice, that measure *D*(*H*,*E*) is undefined whenever $a+c=0$, i.e. when $a=c=0$ (we exclude degenerated cases when $n=0$).
- Exemplary dataset with 6545 different contingency tables (combinations of *a*, *b*, *c* and *d*) contained 33 cases when $a=c=0$.
- Solution: use $ad\text{-}bc>0$ definition of confirmation
  - whenever $ad=bc$, i.e. also when $a=c=0$ assume that $D(H,E)=0$.

Popular confirmation measures

$$Z(H,E)=\begin{cases} 1-\dfrac{\mathrm{P}(\neg H\mid E)}{\mathrm{P}(\neg H)}=\dfrac{ad-bc}{(a+c)(c+d)} & \textit{in case of confirmation} \\[2ex] \dfrac{\mathrm{P}(H\mid E)}{\mathrm{P}(H)}-1=\dfrac{ad-bc}{(a+c)(a+b)} & \textit{in case of disconfirmation} \end{cases}$$

$$A(H,E)=\begin{cases} \dfrac{\mathrm{P}(E\mid H)-\mathrm{P}(E)}{1-\mathrm{P}(E)}=\dfrac{ad-bc}{(a+b)(b+d)} & \textit{in case of confirmation} \\[2ex] \dfrac{\mathrm{P}(H)-\mathrm{P}(H\mid \neg E)}{1-\mathrm{P}(H)}=\dfrac{ad-bc}{(b+d)(c+d)} & \textit{in case of disconfirmation} \end{cases}$$

Derived confirmation measures

$$c_1(H,E) = \begin{cases} \alpha + \beta A(H,E) \ \textit{in case of confirmation when } c = 0 \\ \alpha Z(H,E) \ \textit{in case of confirmation when } c > 0 \\ \alpha Z(H,E) \ \textit{in case of disconfirmation when } a > 0 \\ -\alpha + \beta A(H,E) \ \textit{in case of disconfirmation when } a = 0 \end{cases}$$

$$c_2(H,E) = \begin{cases} \alpha + \beta Z(H,E) \ \textit{in case of confirmation when } b = 0 \\ \alpha A(H,E) \ \textit{in case of confirmation when } b > 0 \\ \alpha A(H,E) \ \textit{in case of disconfirmation when } d > 0 \\ -\alpha + \beta Z(H,E) \ \textit{in case of disconfirmation when } d = 0 \end{cases}$$

$$c_3(H,E) = \begin{cases} A(H,E)Z(H,E) \ \textit{in case of confirmation} \\ -A(H,E)Z(H,E) \ \textit{in case of disconfirmation} \end{cases}$$

$$c_4(H,E) = \begin{cases} \min(A(H,E), Z(H,E)) \ \textit{in case of confirmation} \\ \max(A(H,E), Z(H,E)) \ \textit{in case of disconfirmation} \end{cases}$$

Notation – reminder

- Caution! in the following *c* stands for:
  - $\boldsymbol{c(H,E)}$ – a confirmation measures (general)
  - $\boldsymbol{c_1(H,E)}$, $\boldsymbol{c_2(H,E)}$, $\boldsymbol{c_3(H,E)}$, $\boldsymbol{c_4(H,E)}$ – particular confirmation measures
  - ***c*** – one of the *a*, *b*, *c*, *d* frequencies in the contingency table

Symmetry properties

Symmetry properties

- Symmetry properties are formed by applying
  the negation operator to the rule's premise/conclusion, or both,
  as well as switching the position of the premise and the conclusion.

- Example:

???

$$c(H,E) = c(\neg H, E)$$

$$c(H,E) = c(E, H)$$

$$c(H,E) = c(\neg E, \neg H)$$

Symmetry properties – Carnap, Eells &Fitelson

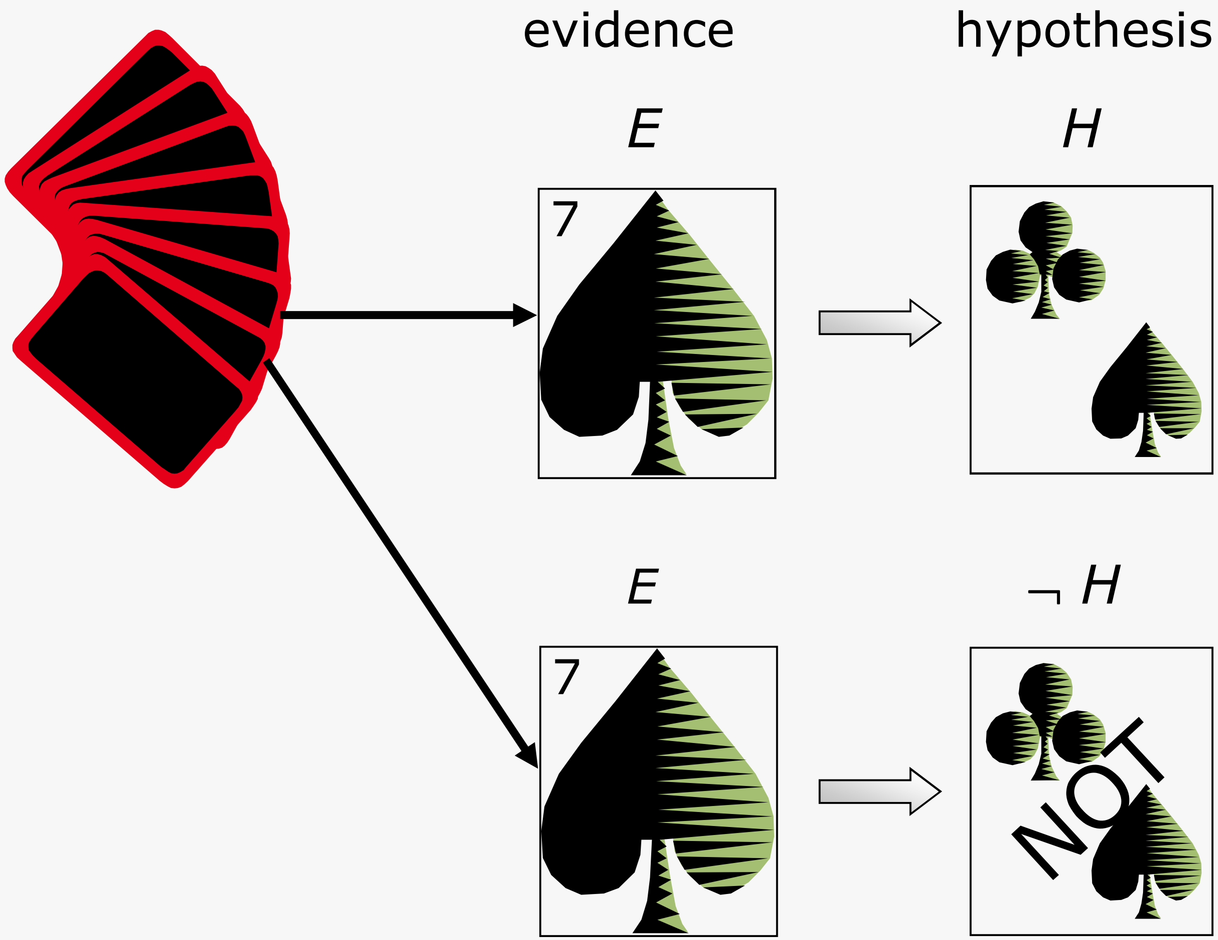
- Carnap, Eells and Fitelson have analyzed confirmation measures from the viewpoint of four properties of symmetry
  - evidence symmetry *ES*: $c(H, E) = -c(H, \neg E)$
  - hypothesis symmetry *HS*: $\boldsymbol{c(H, E) = -c(\neg H, E)}$
  - inversion(commutativity) symmetry *IS*: $c(H, E) = c(E, H)$
  - evidence-hypothesis (total) symmetry *EHS*: $c(H, E) = c(\neg H, \neg E)$

- Their conclusion: only hypothesis symmetry *HS* is a desirable property

Carnap, R., 1962. Logical Foundations of Probability, Univ. of Chicago Press, Chicago.

Eells, E., Fitelson, B., 2002. Symmetries and asymmetries in evidential support. Philosophical Studies, 107 (2): 129-142.

Hypothesis Symmetry (HS)



E is conclusive for *H*

$P(H|E)=1$

(„*the card is not black*")

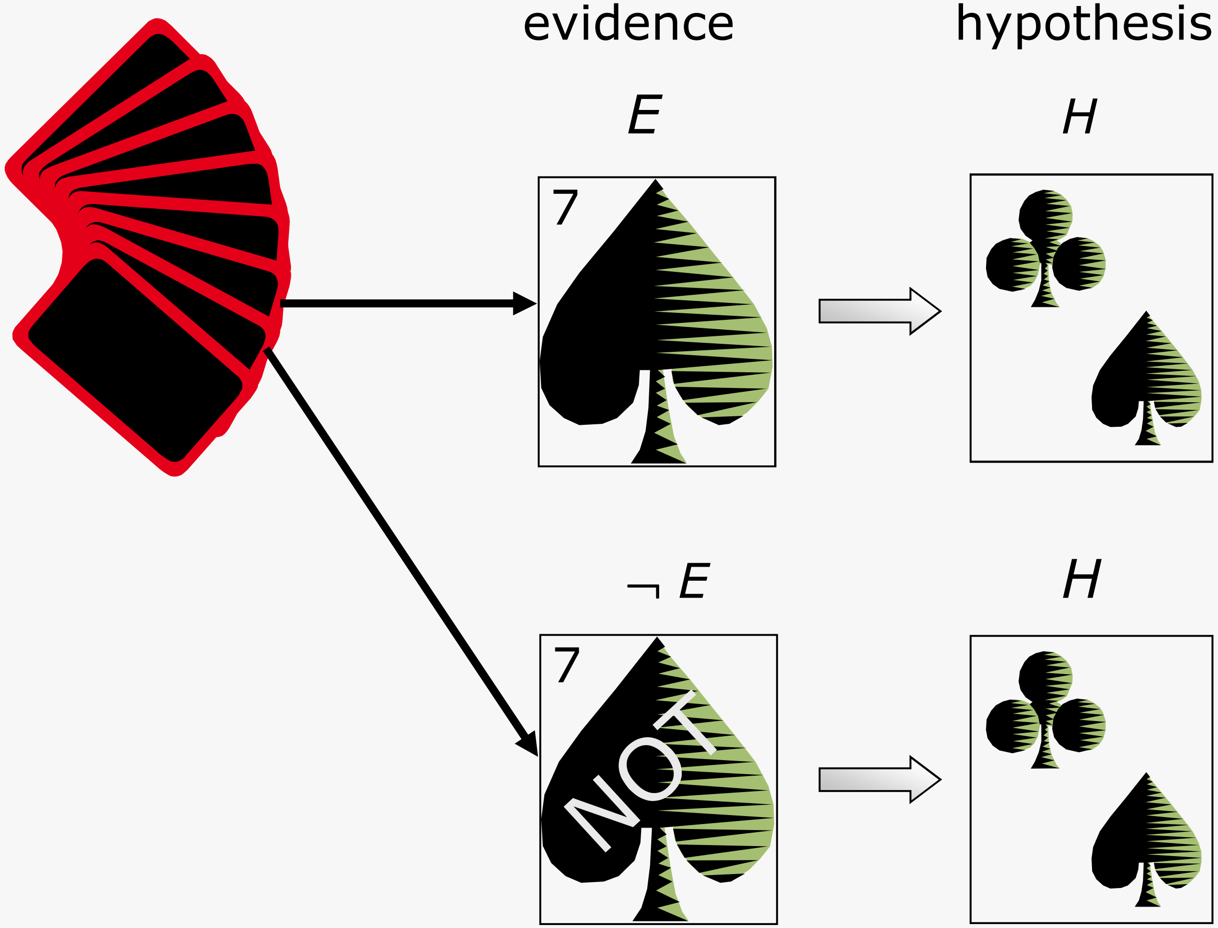
E is negatively conclusive for ¬ *H*
$P(\neg H|E)=0$

$$c(H,E) = -c(\neg H, E)$$

Evidence Symmetry (ES)

evidence — hypothesis („*the card is black*”)

E — H



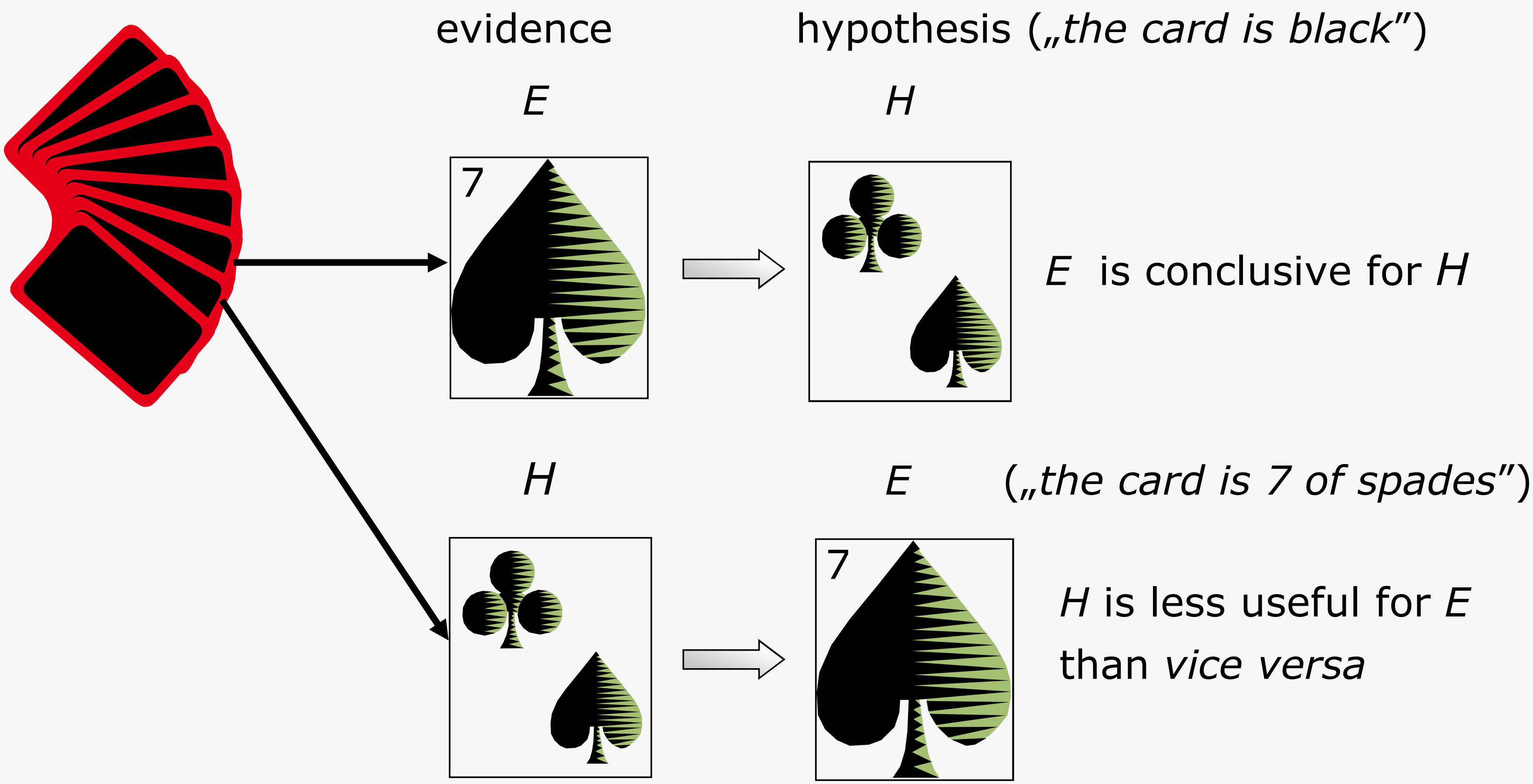
$\neg E$ — H

E is conclusive for H

$\neg E$ is useless for H

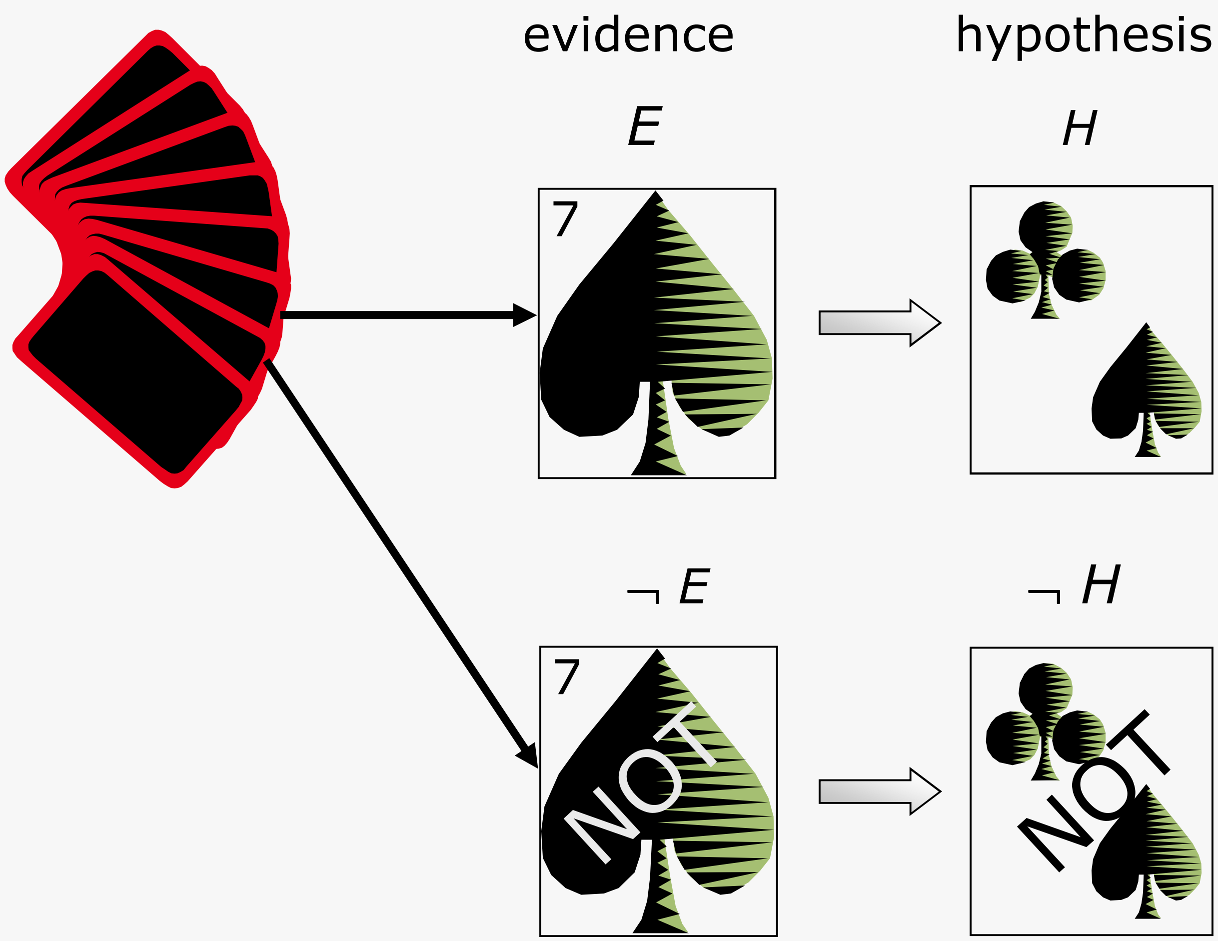
$$c(H,E) = - c(H, \neg E)$$

Inversion Symmetry (IS)



$$c(H,E) = c(E, H)$$

Evidence-hypothesis Symmetry (EHS)



E is conclusive for *H*

¬ *E* is not conclusive for ¬ *H*

$$c(H,E) = - c(\neg H, \neg E)$$

Symmetry properties - Crupi et al.

- Recently, Crupi, Tentori and Gonzalez propose to analyze a confirmation measure $c(H, E)$ with respect to the following symmetries

  $ES(H, E)$: $c(H, E) = -c(H, \neg E)$ $EIS(H, E)$: $c(H, E) = -c(\neg E, H)$

  $HS(H, E)$: $c(H, E) = -c(\neg H, E)$ $HIS(H, E)$: $c(H, E) = -c(E, \neg H)$

  $IS(H, E)$: $c(H, E) = c(E, H)$ $EHIS(H, E)$: $c(H, E) = c(\neg E, \neg H)$

  $EHS(H, E)$: $c(H, E) = c(\neg H, \neg E)$

- Crupi et al. claim that the analysis should be conducted separately for:
  - the case of confirmation (i.e. when $P(H|E) > P(H)$), and
  - for the case of disconfirmation (i.e. when $P(H|E) < P(H)$)
- Such approach results in 14 symmetry properties

Crupi, V., Tentori, K., Gonzalez, M. ,2007. On Bayesian measures of evidential support: Theoretical and empirical issues, *Philosophy of Science*, vol. 74, 229-252.

Crupi et al. symmetries – inversion symmetry

- Crupi et al. concur with the results of Eells and Fitelson regarding the inversion symmetry only in case of confirmation
- Crupi et al. claim that *IS* is desirable in case of disconfirmation
- Let us consider a rule:

  if the drawn card is an Ace, then it is a face

  - the strength with which an *Ace* disconfirms *face* is the same as the strength with which the *face* disconfirms an *Ace*, i.e. $c(H, E) = c(E, H)$
- Conclusions of Crupi et al.:
  - in case of confirmation only the *HS*, *HIS* and *EHIS* are the desirable properties
  - in case of disconfirmation only *HS*, *EIS* and *IS* properties are the desirable properties

Symmetries for Bayesian confirmation - doubts

- The propositions of Eells and Fitelson as well as Crupi et al. are dedicated for the definition of the Bayesian confirmation: $P(H|E)>P(H)$
- Their reasoning is based on assumption that:
  - the highest confirmation should occur in case of entailment ($E|=H \Leftrightarrow P(H|E)=1 \Leftrightarrow c=0$)
  - the highest disconfirmation should occur in case of refutation ($E|=\neg H \Leftrightarrow P(H|E)=0 \Leftrightarrow a=0$)

- Such reasoning boils down to verification whether $P(H|E)$ is 1 or 0
- However the definition of Bayesian confirmation also takes into account $P(H)$.
- Confirmation measures should somehow express: what is the „gain" for *H* from knowing that *E* occured. We want to know if passing from $P(H)$ to $P(H|E)$ is profitable or not.

Symmetries for Bayesian confirmation - doubts

- We want to know if passing from P(H) to P($H|E$) is profitable or not.
- The biggest profits when
  - P(H) is minimal and
  - P($H|E$)=$a/(a+c)=1 \Leftrightarrow c=0$.
- Practical problem: determination when P(H)=(a+b)/n is minimal
  - we want to have a case of confirmation ($ad>bc$), so at least $a\neq 0$
  - P(H) $\rightarrow$ 0 when $n \rightarrow \infty$,
    but we have a closed world of a decision table
- Solution: use the definition of strong Bayesian confirmation
  P($H|E$)>P($H|\neg E$)

Strong Bayesian confirmation

- The definition of strong Bayesian confirmation $P(H|E)>P(H|\neg E)$
  - Confirmation measures should express:
    what is the „gain"(profit) for *H* from passing from $\neg E$ to *E*
  - The biggest profits when:
    - $P(H|\neg E) = 0$ (i.e., $b/(b+d)=0 \Leftrightarrow b=0$) and
    - $P(H|E)=1$ (i.e., $a/(a+c)=1 \Leftrightarrow c=0$).

A new set of symmetries for strong Bayesian confirmation

- A confirmation measure should give an account of the credibility that *it is more probable to have the conclusion when the premise is present, rather than when the premise is absent*
- Both conditional probabilities $P(H|E)$ and $P(H|\neg E)$ should be considered both in case of confirmation and disconfirmation
- There is no need to treat case of confirmation and disconfirmation separately

A new set of symmetries for strong Bayesian confirmation

- *ES*: $c(H, E) = -c(H, \neg E)$ is desirable for strong Bayesian confirmation ($P(H|E) > P(H|\neg E)$)

- ***E* → *H*:** *if the drawn card is the* **7♠**, *then the card is* ***black***

	H	$\neg H$
E	$a=1$	$c=0$
$\neg E$	$b=25$	$d=26$

- Let us observe, that for $c(H, E)$ we have that
  $P(H|\neg E) = b/(b+d) = 25/51 = 0.49$ and
  $P(H|E) = a/(a+c) = 1$, which gives us a 49% increase of confirmation

- On the other hand, for $c(H, \neg E)$ we get:
  $P(H|\neg\neg E) = P(H|E) = 1$ and
  $P(H|\neg E) = 0.49$, which results in 49% decrease

- Thus, clearly the confirmation of a rule $E \rightarrow H$ should be of the same value but of the opposite sign as the confirmation of a $\neg E \rightarrow H$ rule

A new set of symmetries for strong Bayesian confirmation

- *ES*: $c(H, E) = -c(H, \neg E)$ is desirable
- Let us examine both sides of this equation using an exemplary scenario where the values of contingency table of *E* and *H* are:

	H	¬ *H*
E	*a=100*	*c=0*
¬ *E*	*b=10*	*d=40*

- Let us observe, that for $c(H, E)$ we have that

P(H|¬E)= b/(b+d)=0.20 and

P(H|E)=a/(a+c)=1, which gives us a 80% increase of confirmation

- On the other hand, for $c(H, \neg E)$ we get:

P(H|E)=1 and

P(H|¬E)=0.20, which results in 80% decrease

- Thus, clearly the confirmation of a rule $E \rightarrow H$ should be of the same value but of the opposite sign as the confirmation of a $\neg E \rightarrow H$ rule

A new set of symmetries for strong Bayesian confirmation

ES	YES for any (H,E) $c(H, E) = - c(H, \neg E)$
HS	YES for any (H,E) $c(H, E) = - c(\neg H, E)$
EIS	NO for some (H,E) $c(H, E) \neq - c(\neg E, H)$
HIS	NO for some (H,E) $c(H, E) \neq - c(E, \neg H)$
IS	NO for some (H,E) $c(H, E) \neq c(E, H)$
EHS	YES for any (H,E) $c(H, E) = c(\neg H, \neg E)$
EHIS	NO for some (H,E) $c(H, E) \neq c(\neg E, \neg H)$

Symmetries for different definitions of confirmation

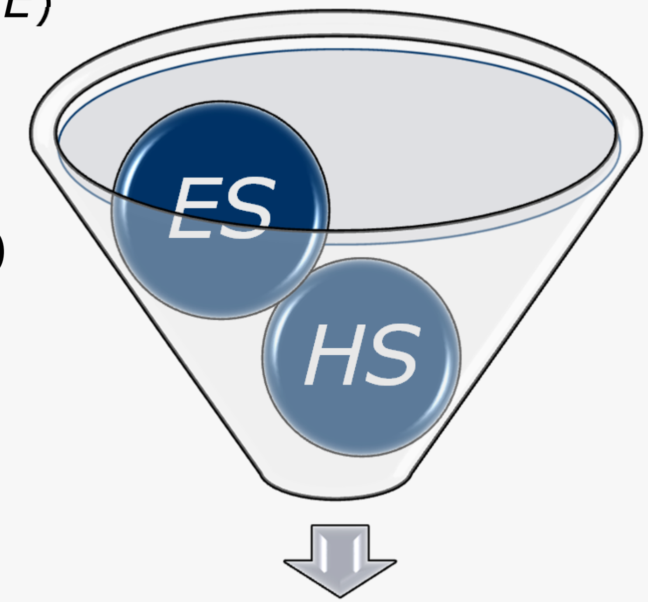
- Systematic approach to symmetry properties in the context of different definitions of confirmation
- Let us focus only on the probabilities involved in different definitions of confirmation:
  a) $P(H|E)$ and $P(H)$ for Bayesian confirmation
  b) $P(H|E)$ and $P(H|\neg E)$ for strong Bayesian confirmation
  c) $P(E|H)$ and $P(E)$ for likelihoodist confirmation
  d) $P(E|H)$ and $P(E|\neg H)$ for strong likelihoodist confirmation

Symmetries for different definitions of confirmation

- From basic probability theory:
  a) for Bayesian confirmation $P(H|E)>P(H)$:
  $P(\neg H|E)=1-P(H|E)$ and
  $P(\neg H)=1-P(H)$;
  - hypothesis symmetry: $c(H,E)=-c(\neg H,E)$
    because $P(\neg H|E)<P(\neg H)$ is equivalent to $P(H|E)> P(H)$

Symmetries for different definitions of confirmation

- From basic probability theory:
  - b) for strong Bayesian confirmation $P(H|E) > P(H|\neg E)$:
    $P(\neg H|E)=1-P(H|E)$ and
    $P(\neg H|\neg E)=1-P(H|\neg E)$;
  - hypothesis symmetry: $c(H,E)=-c(\neg H,E)$
    because $P(\neg H|E)<P(\neg H|\neg E)$
    is equivalent to $P(H|E)> P(H|\neg E)$
  - evidence symmetry: $c(H,E)=-c(H,\neg E)$
    because $P(H|\neg E)<P(H|E)$
    is equivalent to $P(H|E)< P(H|\neg E)$
  - evidence-hypothesis symmetry:



EHS: $c(H,E)=c(\neg H,\neg E)$

Symmetries for different definitions of confirmation - summary

Definition of confirmation	Desirable symmetry
Bayesian confirmation	HS
strong Bayesian confirmation	ES, HS, EHS
likelihoodist confirmation	ES
strong likelihoodist confirmation	ES, HS, EHS

Property of concordance

Thank you!